III. REPRESENTATIONS OF PHOTON STATES

1. Fock or "Number" States: .[11]

As we have seen, the Fock or *number* states

$$\left|\left\{ n_{\vec{\mathbf{k}}} \right\}\right\rangle \equiv \prod_{s\,\sigma} \left| n_{\vec{\mathbf{k}}_s\,\sigma} \right\rangle \qquad \text{[III-1]}$$

are complete set eigenstates of an important group of commuting observables -- *viz.* $\mathcal{H}_{rad}$, $\mathcal{N}$ and $\vec{\mathcal{M}}$.

Reprise of Characteristics and Properties of Fock States:

a. **The expectation value of the number operator and the *fractional uncertainty* associated with a single Fock state**:

$$\langle n | \mathcal{N} | n \rangle = n \qquad \text{[III-2a]}$$

$$\Delta n = [\ \text{"uncertainty"}\ ] = \sqrt{\left\{ \langle n | \mathcal{N}^2 | n \rangle - \langle n | \mathcal{N} | n \rangle^2 \right\}} = 0 \qquad \text{[III-2b]}$$

b. **Expectation value of the fields associated with a single mode**:

For one mode Equations [II-24a] and [II-24b] reduce to

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = i\,\hat{\mathbf{e}}\,\mathcal{E} \left\{ a \exp\left[i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - i\,\omega\,t \right] - a^{\dagger} \exp\left[-i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - i\,\omega\,t \right] \right\} \qquad \text{[III-3a]}$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}},t) = i\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\mathcal{E}\left[\hat{\mathbf{k}}\times\hat{\mathbf{e}} \right] \left\{ a(t) \exp\left[i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - i\,\omega\,t \right] - a^{\dagger}(t) \exp\left[-i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - i\,\omega\,t \right] \right\} \qquad \text{[III-3b]}$$

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[11] In what follows, for simplicity we drop the $\vec{\mathbf{k}}$ subscripts on the operators and state vectors with the obvious meaning that $\left|\left\{ n_{\vec{\mathbf{k}}} \right\}\right\rangle \rightarrow |n\rangle$, $a_{\vec{\mathbf{k}}} \rightarrow a$, *etc...*

where $\mathcal{E} = \sqrt{\dfrac{\hbar\omega}{2\varepsilon_0 V}}$

$$\begin{aligned} \langle n | \vec{\mathbf{E}} | n \rangle &= 0 \\ \langle n | \vec{\mathbf{H}} | n \rangle &= 0 \end{aligned} \qquad \text{[III-4a]}$$

$$\begin{aligned} \Delta\mathbf{E} &= \sqrt{\left\{ \langle n | \vec{\mathbf{E}} \; \vec{\mathbf{E}} | n \rangle - \langle n | \vec{\mathbf{E}} | n \rangle^2 \right\}} = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \left(n + \tfrac{1}{2}\right)^{1/2} = \sqrt{2}\, \mathcal{E} \left(n + \tfrac{1}{2}\right)^{1/2} \\ \Delta\mathbf{H} &= \sqrt{\left\{ \langle n | \vec{\mathbf{H}} \; \vec{\mathbf{H}} | n \rangle - \langle n | \vec{\mathbf{H}} | n \rangle^2 \right\}} = \sqrt{\frac{\hbar\omega}{\mu_0 V}} \left(n + \tfrac{1}{2}\right)^{1/2} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \sqrt{2}\, \mathcal{E} \left(n + \tfrac{1}{2}\right)^{1/2} \\ \Delta\mathbf{E}\, \Delta\mathbf{H} &= c\, \frac{\hbar\omega}{V} \left(n + \tfrac{1}{2}\right) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, 2\, \mathcal{E}^2 \left(n + \tfrac{1}{2}\right) \end{aligned} \qquad \text{[III-4b]}$$

c. **Phase of field associated with single mode**:

To obtain something analogous to the classical theory we would like to separate the creation and destruction operators (and, thus, the electric and magnetic field operators) into a product of amplitude and phase operators. Following Susskind and Glogower,[12] we define a ***phase operator***, Φ such that

$$\begin{aligned} a &\equiv \left(\mathcal{N} + 1\right)^{1/2} \exp(i\,\Phi) \\ a^\dagger &\equiv \exp(-i\,\Phi) \left(\mathcal{N} + 1\right)^{1/2} \end{aligned} \qquad \text{[III-5]}$$

Defined in this way, the basic properties of the phase operator may be evaluated from known properties of the creation, destruction and number operators. Inverting, we obtain

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12 Susskind, L. and Glogower, J., *Physics*, **1**, 49 (1964)

$$\exp(i\,\Phi) \equiv \left(\mathcal{N}+1\right)^{-1/2} a$$
$$\exp(-i\,\Phi) \equiv a^{\dagger}\left(\mathcal{N}+1\right)^{-1/2} \qquad [\text{ III-6 }]$$

and since $a\,a^{\dagger} = \mathcal{N}+1$, it follows that

$$\exp(i\,\Phi)\,\exp(-i\,\Phi) = 1 \qquad [\text{ III-7 }]$$

but only in this order! Operating on number states with the phase operators, we obtain from Equation [I-26]

$$\exp(i\,\Phi)\,|n\rangle = \left(\mathcal{N}+1\right)^{-1/2} a\,|n\rangle = \left(\mathcal{N}+1\right)^{-1/2}(n)^{1/2}\,|n-1\rangle = |n-1\rangle$$
$$\exp(-i\,\Phi)\,|n\rangle = a^{\dagger}\left(\mathcal{N}+1\right)^{-1/2}|n\rangle = a^{\dagger}\,(n+1)^{-1/2}\,|n\rangle = |n+1\rangle \qquad [\text{ III-8 }]$$

Consequently, the **only nonvanishing matrix elements** of the phase operator are

$$\langle n-1|\exp(i\,\Phi)\,|n\rangle = 1$$
$$\langle n+1|\exp(-i\,\Phi)\,|n\rangle = 1 \qquad [\text{ III-9 }]$$

The phase operators defined by Equation [III-36] do have the felicitous or *classically analogous* property of revealing ***magnitude independent*** information, but unfortunately they are nonHermitian operators -- *i.e.*

$$\langle n-1|\exp(i\,\Phi)\,|n\rangle \neq \langle n|\exp(i\,\Phi)\,|n-1\rangle$$

-- and, hence, **cannot represent observables**. However, they may be **paired** into operators that are observables -- *viz*.

$$\cos\Phi = \frac{1}{2}\left\{\exp(i\,\Phi) + \exp(-i\,\Phi)\right\}$$
$$\sin\Phi = \frac{1}{2\,i}\left\{\exp(i\,\Phi) - \exp(-i\,\Phi)\right\} \qquad \text{[III-10]}$$

which have the following nonvanishing matrix elements:

$$\langle n-1|\cos\Phi|n\rangle = \langle n|\cos\Phi|n-1\rangle = \frac{1}{2}$$
$$\langle n-1|\sin\Phi|n\rangle = -\langle n|\sin\Phi|n-1\rangle = \frac{1}{2\,i} \qquad \text{[III-11]}$$

These *nearly commuting* operators[13] may be adopted as the quantum mechanical operators which represent (as we will demonstrate anon) the observable phase properties of the electromagnetic field.

For the Fock state:

$$\langle n|\cos\Phi|n\rangle = \langle n|\sin\Phi|n\rangle = 0 \qquad \text{[III-12a]}$$

$$\Delta\cos\Phi = \Delta\sin\Phi = \sqrt{\left\{\langle n|\cos^2\Phi|n\rangle - \langle n|\cos\Phi|n\rangle^2\right\}} = \sqrt{1/2} \qquad \text{[III-12b]}$$

$$\Delta\cos\Phi\;\Delta\sin\Phi = 1/2 \qquad \text{[III-12c]}$$

c. **The coordinate or Schrödinger representation of state:**

Recall from Equations [I-10a] and [I-31] that

13 Also, it may be easily established that the matrix elements of their commutator are given by

$$\langle n|[\cos\Phi, \sin\Phi]|n'\rangle = \frac{i}{2}\,\delta_{nn'}\,\delta_{n0}$$

$$\langle q|n\rangle = \frac{1}{\sqrt{n!}}\left[\sqrt{\frac{m}{2\hbar\omega}}\right]^n \left[\omega q - \frac{\hbar}{m}\frac{d}{dq}\right]^n \langle q|0\rangle$$

$$= \sqrt{\frac{1}{2^n\, n!}\sqrt{\frac{\omega}{\pi\hbar}}}\; H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, q\right)\exp\left[-\frac{m\omega}{2\hbar}q^2\right] \qquad \text{[III-13]}$$

Therefore, the probability $P(q)$ of eigenvalues q for a given Fock state $|n\rangle$ is give by

$$P(q) = \left|\langle q|n\rangle\right| = \frac{1}{2^n\, n!}\sqrt{\frac{\omega}{\pi\hbar}}\; H_n^2\left(\sqrt{\frac{m\omega}{\hbar}}\, q\right)\exp\left[-\frac{m\omega}{\hbar}q^2\right] \qquad \text{[III-14]}$$

d. **Approximate "localization" of a photon:** [14]

Of course a plane wave is distributed or "de-localized" in both time and space. Defining the "wave function for a photon" is a task fraught with danger,[15] but the simpler task of defining a wave function approximately localized at a given instant is relatively straight forward -- *viz*.

$$\left|\psi(\vec{\mathbf{r}}_0)\right\rangle_{\vec{\mathbf{k}}_0 \left|\Delta\vec{\mathbf{k}}\right|^2} = C\sum_{\vec{\mathbf{k}}}\exp\left[\frac{\left|\vec{\mathbf{k}}-\vec{\mathbf{k}}_0\right|^2}{2\left|\Delta\vec{\mathbf{k}}\right|^2}\right]\exp\left[i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}_0\right]\left|0,0,0,\ldots n_{\vec{\mathbf{k}}}=1,\ldots,0,0,0\right\rangle \qquad \text{[III-14]}$$

2. **<u>Photon States of *Well-defined* Phase</u>:**

Consider the state defined by

$$|\phi\rangle \equiv \lim_{s\to\infty}(s+1)^{-1/2}\sum_{n=0}^{s}\exp[i\,n\,\phi]\,|n\rangle \qquad \text{[III-15]}$$

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14 See Section 10.4.2 in Leonard Mandel and Emil Wolf, *Optical Coherence and Quantum Optics,* Cambridge Press (1995), ISBN 0-521-417112.

15 See Section 1.5.4 in Marlan O. Scully and M. Suhail Zubairy, Quantum Optics, Cambridge Press (1997), ISBN 0-521-43458.

Clearly, $\langle \phi | \phi \rangle = 1$ given the orthonormal properties of the number states. Essential question: Is this state an eigenstate of the phase operators? To answer the question we need to consider the following ***potential eigenvalue equation***:

$$\cos\Phi\,|\phi\rangle = \frac{1}{2}\lim_{s\to\infty}\left(s+1\right)^{-1/2}\left\{\sum_{n=0}^{s}\exp\left[i\,n\,\phi\right]\,\exp\left[i\,\Phi\right]|n\rangle + \sum_{n=0}^{s}\exp\left[i n\,\phi\right]\,\exp\left[-i\,\Phi\right]|n\rangle\right\} \quad \text{[III-16a]}$$

Using Equations [III-10] and [III-10], we obtain

$$\begin{aligned}
\cos\Phi\,|\phi\rangle &= \frac{1}{2}\lim_{s\to\infty}\left(s+1\right)^{-1/2}\left\{\sum_{n=1}^{s}\exp\left[i n\,\phi\right]|n-1\rangle + \sum_{n=0}^{s}\exp\left[i\,n\,\phi\right]|n+1\rangle\right\} \\
&= \frac{1}{2}\lim_{s\to\infty}\left(s+1\right)^{-1/2}\left\{\exp\left(i\phi\right)\sum_{\nu=0}^{s-1}\exp\left[i\,\nu\,\phi\right]|\nu\rangle + \exp\left[-i\,\phi\right]\sum_{\nu=1}^{s+1}\exp\left[i\,\nu\,\phi\right]|\nu\rangle\right\} \quad \text{[III-16b]} \\
&= \cos\phi\,|\phi\rangle \\
&\quad + \frac{1}{2}\lim_{s\to\infty}\left(s+1\right)^{-1/2}\left\{\exp\left[i\,s\,\phi\right]|s+1\rangle - \exp\left[i\left(s+1\right)\phi\right]|s\rangle - \exp\left[-i\,\phi\right]|0\rangle\right\}
\end{aligned}$$

so that the state $|\phi\rangle$ fails to be a strict eigenket of $\cos\Phi$ by terms that diminish faster than $\left(s+1\right)^{-1/2}$ as $s\to\infty$. Similarly, we can see that diagonal matrix elements of $\cos\Phi$ and $\sin\Phi$ are given by

$$\langle\phi|\cos\Phi|\phi\rangle = \cos\phi\left\{1 - \lim_{s\to\infty}\left(s+1\right)^{-1}\right\} \qquad \cos\phi \qquad \text{[III-17a]}$$

$$\langle\phi|\sin\Phi|\phi\rangle = \sin\phi\left\{1 - \lim_{s\to\infty}\left(s+1\right)^{-1}\right\} \qquad \sin\phi \qquad \text{[III-17b]}$$

Reprise of Characteristics and Properties of Phase States:

a. **The expectation value of the number operator and the *fractional uncertainty* associated with a state of well-defined phase**:

$$\langle\phi|\mathcal{N}|\phi\rangle = \lim_{s\to\infty}(s+1)^{-1}\sum_{n=0}^{s} n = \lim_{s\to\infty}(s+1)^{-1}\left[\frac{s(s+1)}{2}\right] = \lim_{s\to\infty}\frac{s}{2} \qquad \text{[III-18a]}$$

$$\begin{aligned}\left[\begin{matrix}\text{fractional}\\ \text{uncertainty}\end{matrix}\right] &= \frac{\sqrt{\left\{\langle\phi|\mathcal{N}^2|\phi\rangle-\langle\phi|\mathcal{N}|\phi\rangle^2\right\}}}{\langle\phi|\mathcal{N}|\phi\rangle}\\ &= \frac{\sqrt{\left\{\lim_{s\to\infty}(s+1)^{-1}\sum_{n=0}^{s}n^2-\left[\lim_{s\to\infty}(s+1)^{-1}\sum_{n=0}^{s}n\right]^2\right\}}}{\lim_{s\to\infty}(s+1)^{-1}\sum_{n=0}^{s}n}\\ &= \frac{\sqrt{\left\{\lim_{s\to\infty}\left[\frac{1}{6}\left(2s^2+s\right)-\frac{1}{4}s^2\right]\right\}}}{\lim_{s\to\infty}\frac{s}{2}} = \frac{1}{\sqrt{3}}\end{aligned} \qquad \text{[III-18b]}$$

b. **Expectation value of the fields associated with a single mode**:

From Equation [III-3a]

$$\langle\phi|\vec{\mathbf{E}}|\phi\rangle = -2\sqrt{\frac{\hbar\,\omega}{2\,\varepsilon_0\,V}}\,\hat{\mathbf{e}}\,\sin\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega\,t+\phi\right)\lim_{s\to\infty}(s+1)^{-1}\sum_{n=0}^{s}(n+1)^{1/2} \qquad \text{[III-19]}$$

diverges as $\sqrt{s}$ for large s !

c. **Phase of field associated with single mode**:

$$\langle \phi | \cos\Phi | \phi \rangle = \cos\phi$$
$$\langle \phi | \sin\Phi | \phi \rangle = \sin\phi \qquad [\text{ III-20a }]$$

$$\Delta\cos\Phi = \Delta\sin\Phi = \sqrt{\left\{ \langle \phi | \cos^2\Phi | \phi \rangle - \langle \phi | \cos\Phi | \phi \rangle^2 \right\}} = 0 \qquad [\text{ III-20b }]$$

d. **Probability of photon number**:

Finally, we may easily deduce the probability of finding n photons (*i.e.* the photon statistics) in a particular state of well defined phase -- *viz.*

$$P_n = \left| \langle n | \phi \rangle \right|^2 \equiv \lim_{s \to \infty} (s+1)^{-1} \qquad [\text{ III-50 }]$$

We see that there is a equal, but small probability of any number: this agrees with the intuition that the magnitude of the field is completely undetermined if the phase is precisely known!

3. ***Coherent* Photon States:** [16]

It would, indeed, be useful to have eigenstates of the ***destruction operator*** (electric or magnetic field) -- *viz.*

$$a_{\vec{k}} \left| \alpha_{\vec{k}} \right\rangle = \alpha_{\vec{k}} \left| \alpha_{\vec{k}} \right\rangle \qquad [\text{ III-51 }]$$

Reprise of Characteristics and Properties of Coherent States:

a. **The Fock state representation of the coherent state**:

[16] The coherent state is a **Harvard invention**! See R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

Since. $a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $a\,a^{\dagger} = \mathcal{N} + 1$, then $\langle n|\,a = \sqrt{n+1}\,\langle n+1|$ and we are able to write a **representative** of the sought state in the number state basis -- *viz*.

$$\langle n|\,a\,|\alpha\rangle = \sqrt{n+1}\,\langle n+1|\alpha\rangle = \alpha\,\langle n|\alpha\rangle \qquad \text{[III-52a]}$$

or

$$\langle n|\alpha\rangle = \frac{\alpha}{\sqrt{n}}\langle n-1|\alpha\rangle = \frac{\alpha^{n}}{\sqrt{n!}}\langle 0|\alpha\rangle \qquad \text{[III-52b]}$$

Using the expansion of the identity operator, the eigenket becomes

$$|\alpha\rangle = \sum_{n}|n\rangle\langle n|\alpha\rangle = \langle 0|\alpha\rangle \sum_{n}\frac{\alpha^{n}}{\sqrt{n!}}\,|n\rangle\ . \qquad \text{[III-53]}$$

To normalize the eigenket write

$$\langle\alpha|\alpha\rangle = \langle\alpha|0\rangle\langle 0|\alpha\rangle \sum_{n}\frac{\alpha^{*\,n}\,\alpha^{n}}{n!} = \langle\alpha|0\rangle\langle 0|\alpha\rangle\,\exp\left[|\alpha|^{2}\right] = 1 \qquad \text{[III-54]}$$

so that $\langle\alpha|0\rangle = \langle 0|\alpha\rangle = \exp\left[\left(-\frac{1}{2}|\alpha|^{2}\right)\right]$. Finally, we see that

$$|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^{2}\right]\sum_{n}\frac{\alpha^{n}}{\sqrt{n!}}\,|n\rangle \qquad \text{[III-55]}$$

is a normalize representation of the eigenkets of the destruction operator.

b. **The expectation value of the number operator and the *fractional uncertainty* associated with a coherent state**:

$$\langle \alpha | \mathcal{N} | \alpha \rangle = |\alpha|^2 \qquad \text{[III-56a]}$$

$$\begin{bmatrix} \text{fractional} \\ \text{uncertainty} \end{bmatrix} = \frac{\sqrt{\left\{ \langle \alpha | \mathcal{N}^2 | \alpha \rangle - \langle \alpha | \mathcal{N} | \alpha \rangle^2 \right\}}}{\langle \alpha | \mathcal{N} | \alpha \rangle} = \frac{1}{|\alpha|^2} \sqrt{\left\{ \exp\left(-|\alpha|^2\right) \sum \frac{|\alpha|^{2n}}{n!} n^2 - |\alpha|^4 \right\}}$$

$$= \frac{1}{|\alpha|^2} \sqrt{\left\{ \exp\left(-|\alpha|^2\right) \sum \frac{|\alpha|^{2n}}{n!} \left[n\left(n-1\right) + n \right] - |\alpha|^4 \right\}} \qquad \text{[III-56b]}$$

$$= |\alpha|^{-1}$$

Thus, we see that the fractional uncertainty diminishes with mean photon number!

c. **Expectation value of the electric field associated with a single mode**:

From Equation [III-3a]

$$\langle \alpha | \mathbf{E} | \alpha \rangle = -2 \sqrt{\frac{\hbar \omega}{2 \varepsilon_0 V}} \, \hat{\mathbf{e}} \, |\alpha| \sin\left(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t + \vartheta \right) \qquad \text{[III-57a]}$$

where $\alpha = |\alpha| \exp\left(i \vartheta \right)$.

$$\Delta \mathbf{E} = \sqrt{\left\{ \langle \alpha | \vec{\mathbf{E}} \cdot \vec{\mathbf{E}} | \alpha \rangle - \langle \alpha | \vec{\mathbf{E}} | \alpha \rangle^2 \right\}} = \sqrt{\frac{\hbar \omega}{2 \varepsilon_0 V}} \text{ [17]} \qquad \text{[III-57b]}$$

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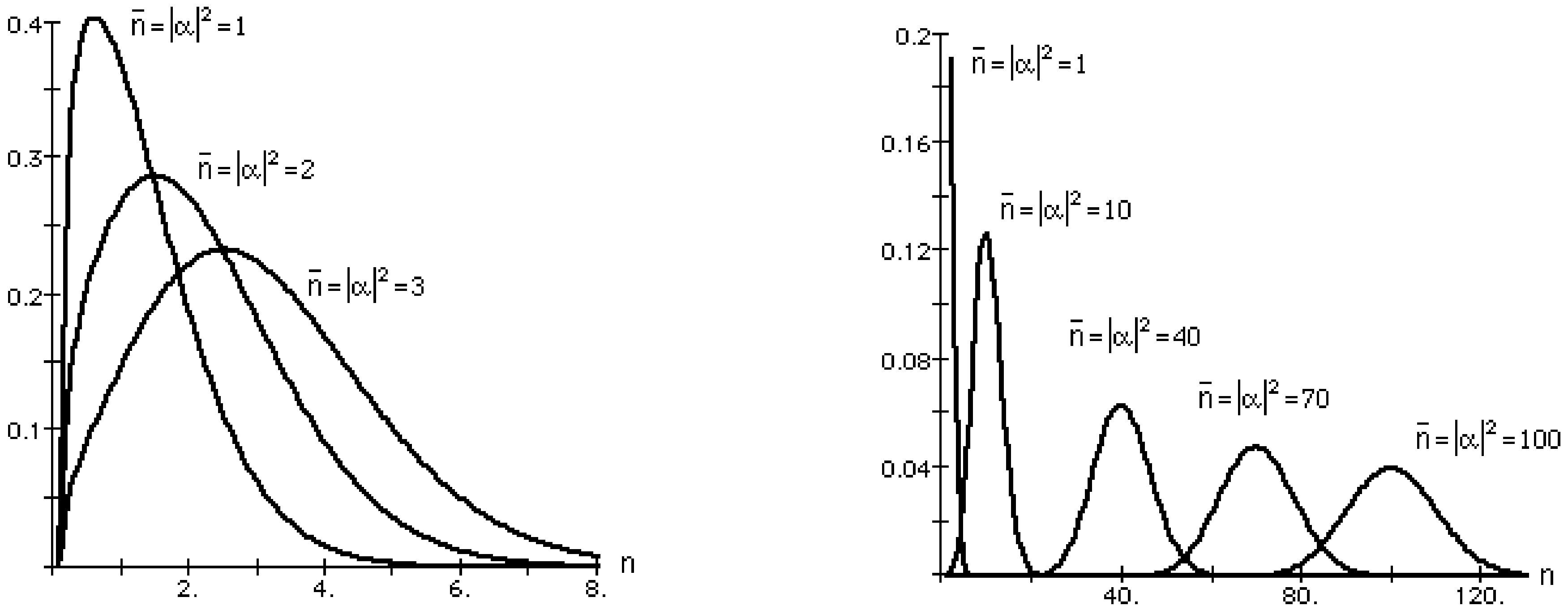
[17] Similarly $\Delta \mathbf{H} = \dfrac{1}{c \, \mu_0} \sqrt{\dfrac{\hbar \omega}{2 \varepsilon_0 V}}$ for the coherent state, so that $\Delta \mathbf{E} \, \Delta \mathbf{H} = c \, \hbar \, \omega / 2 \, V$.

d. **Probability of photon number**:

From the representation of the coherent state given in Equation [III-55] we may easily deduce the probability of finding *n* photons (the photon statistics) in a particular coherent state is given by a **Poisson distribution** characterized by the mean value $\bar{n} = |\alpha|^2$. -- *viz.*

$$P_n = |\langle n | \alpha \rangle|^2 = \exp\left[-|\alpha|^2\right] \frac{|\alpha|^{2n}}{n!} \qquad \text{[III-58]}$$

SAMPLE POISSON DISTRIBUTIONS - COHERENT STATE PHOTON STATISTICS



e. **Phase of field associated with single mode**:

$$
\begin{aligned}
\langle\alpha\,|\cos\Phi|\,\alpha\rangle &= \frac{1}{2}\exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{n'}\sum_{n}\langle n'|\frac{\alpha^{*n'}}{\sqrt{n'!}}\left[(\mathcal{N}+1)^{-1/2}\,a + a^{\dagger}\,(\mathcal{N}+1)^{-1/2}\right]\frac{\alpha^{n}}{\sqrt{n!}}|n\rangle \\
&= \frac{1}{2}\exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{n}\left[\frac{\alpha^{*n+1}\,\alpha^{n} + \alpha^{*n}\,\alpha^{n+1}}{\sqrt{\{(n+1)!\,n!\}}}\right] \\
&= |\alpha|\cos\vartheta\,\exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{n}\left[\frac{|\alpha|^{2n}}{n!\,\sqrt{(n+1)}}\right]
\end{aligned}
\qquad [\text{ III-59a}]
$$

Unfortunately, it is not possible to evaluate this summation analytically. However, Carruthers[18] has given an asymptotic expansion which is valid for a large mean number of photons -- *viz*.

$$
\langle\alpha\,|\cos\Phi|\,\alpha\rangle = \cos\vartheta\left[1 - \frac{1}{8|\alpha|^2} + \ldots\right] \qquad |\alpha|^2 >> 1 \qquad [\text{ III-59b}]
$$

f. **Coherent states as a basis:**

As we will see presently, the coherent states are very useful in describing the quantized electromagnetic field, but, alas, there is a complication -- **the coherent states are not truly orthogonal!** From Equation [III-6] we see that

$$
\begin{aligned}
\langle\beta|\alpha\rangle &= \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2\right]\sum_{n}\frac{\beta^{*n}\alpha^{n}}{n!} \\
&= \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha\,\beta^{*}\right]
\end{aligned}
\qquad [\text{ III-60 }]
$$

so that

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[18] Carruthers, P. and Nieto, M. M., *Phys. Rev. Lett.* **14**, 387 (1965)

$$
\begin{aligned}
\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle &= \exp\left(-|\alpha|^2-|\beta|^2+\alpha^*\beta+\alpha\beta^*\right) \\
&= \exp\left(-(\alpha-\beta)\left(\alpha^*-\beta^*\right)\right) = \exp\left(-|\alpha-\beta|^2\right)
\end{aligned}
\qquad \text{[III-61]}
$$

That is, t**he eigenkets are approximately orthogonal** only when $|\alpha-\beta|$ is large!

g. **The "displacement operator:"**

There are a growing and significant set of applications where it is useful to express the coherent states directly in terms of the vacuum state $|0\rangle$. If we use the number state generating rule

$$
|n\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0\rangle
$$

-- *i.e.* Equation [I-27] -- the coherent state may be written in the form

$$
|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_n \frac{\left(\alpha\, a^\dagger\right)^n}{n!}|0\rangle = \exp\left(\alpha\, a^\dagger - \frac{1}{2}|\alpha|^2\right)|0\rangle \qquad \text{[III-62]}
$$

If we make us of the Baker-Hausdorff theorem,[19] we may easily show that

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19 The Baker-Hausdorff theorem or identity may be stated as

$$
\exp\{\mathcal{A}+\mathcal{B}\} = \exp\{\mathcal{A}\}\,\exp\{\mathcal{B}\}\,\exp\left\{-\tfrac{1}{2}[\mathcal{A},\mathcal{B}]\right\}
$$

when $[\mathcal{A},[\mathcal{A},\mathcal{B}]] = [\mathcal{B},[\mathcal{A},\mathcal{B}]] = 0$. For a proof, see, for example, Charles P. Slichter's *Principles of Magnetic Resonance*, Appendix A or William Louisell's *Radiation and Noise in Quantum Electronics*.

$$|\alpha\rangle = \mathcal{A}^{\dagger}(\alpha)\,|0\rangle = \exp\left(\alpha\, a^{\dagger} - \alpha\;\; a\right)|0\rangle \qquad [\text{ III-63 }]$$

so that $\mathcal{A}^{\dagger}(\alpha)$ may be interpreted as a *creation* operator which generates a coherent state from the vacuum. (Its adjoint operator $\mathcal{A}(\alpha) = \mathcal{A}^{\dagger}(-\alpha)$ is a *destruction* operator which destroys a state). In some treatments $\mathcal{A}^{\dagger}(\alpha)$ is described as the "displacement operator" (written $\mathcal{D}(\alpha)$)[20] and the coherent states are called the "displaced states of the vacuum." [21]

To explore this point of view (and to give some meaning to the phase of the coherent state eigenvalue), we may express $|\alpha\rangle$ in a two-dimensional, dimensionless "phase space" representation. To that end, following Equation [I-16], we write the dimensionless coordinate as

$$\theta = \left(\frac{m\,\omega}{\hbar}\right)^{1/2} q = a^{\dagger}\exp[i\,\gamma] + a\;\;\exp[-i\,\gamma] \qquad [\text{ III-64a }]$$

and the dimensionless momentum as

$$\pi = \left(\frac{2}{m\,\hbar\,\omega}\right)^{1/2} p = a^{\dagger}\exp\left[i\left(\gamma + \pi/2\right)\right] + a\;\;\exp\left[-i\left(\gamma + \pi/2\right)\right] \qquad [\text{ III-64b }]$$

so that

$$[\theta, \pi] = 2\,i\left[a\,, a^{\dagger}\right] = 2\,i \qquad [\text{ III-64c }]$$

---

20 We can (or rather you will) show that $\mathcal{D}^{\dagger}(\alpha)\, a\;\mathcal{D}(\alpha) = a + \alpha$ and $\mathcal{D}^{\dagger}(\alpha)\, a^{\dagger}\,\mathcal{D}(\alpha) = a^{\dagger} + \alpha$

21 See *Elements of Quantum Optics*, Pierre Meystre and Murray Sargent III, Spinger-Verlag (1991), ISBN 0-387-54190-X.

and since these variables are canonical [22]

$$\left\langle (\Delta\theta)^2 \right\rangle \left\langle (\Delta\pi)^2 \right\rangle \geq 1 \qquad \text{[III-64d]}$$

Since

$$\begin{aligned} a^\dagger &= \frac{1}{2}(\theta - i\,\pi)\,\exp[-i\,\gamma] \\ a &= \frac{1}{2}(\theta + i\,\pi)\,\exp[i\,\gamma] \end{aligned} \qquad \text{[III-65]}$$

the mode field (see Equation [II-24a]) b

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = i\,\hat{\mathbf{e}}\,\mathcal{E}\left\{ a\ \exp\left[i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - i\,\omega\,t\right] - a^\dagger\ \exp\left[-i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} + i\,\omega\,t\right]\right\} \qquad \text{[III-66a]}$$

becomes

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = -\,\hat{\mathbf{e}}\,\mathcal{E}\left\{\pi\,\cos\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega\,t + \gamma\right) + \theta\,\sin\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega\,t + \gamma\right)\right\} \qquad \text{[III-66b]}$$

Since p has a coordinate space representation $-i\,\hbar\,d/dq = -i\left(\hbar\,\omega/2\right)^{1/2}\,d/d\theta$ and q has a momentum representation $i\,\hbar\,d/d\,p = i\left(\hbar/2\,\omega\right)^{1/2}\,d/d\pi$, [23]

$$\begin{aligned} \alpha\,a^\dagger - \alpha\ a &= \alpha_r\left(a^\dagger - a\right) + i\,\alpha_i\left(a^\dagger + a\right) \\ &= -\left[\alpha_r\ d/d\theta + \alpha_i\ d/d\pi\right] \end{aligned} \qquad \text{[III-67a]}$$

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22 Of course, in general $\left\langle (\Delta\mathbf{A})^2 \right\rangle \left\langle (\Delta\mathbf{B})^2 \right\rangle \geq \frac{1}{2}\left|\left\langle [\ \mathbf{A},\mathbf{B}\ ] \right\rangle\right|^2$ where $\left\langle (\Delta\mathbf{A})^2 \right\rangle = \left\langle \mathbf{A}^2 \right\rangle - \left\langle \mathbf{A} \right\rangle^2$

23 If this unfamiliar, see Equations [I-20] and [I-22] in the lecture notes entitled *The Interaction of Radiation and Matter: Semiclassical Theory*.

and

$$\mathcal{A}^{\dagger}(\alpha) = \exp\left(\alpha\, a^{\dagger} - \alpha^{*}\, a\right) = \exp\left[-\left(\alpha_r\, d/d\theta + \alpha_i\, d/d\pi\right)\right] \qquad \text{[III-67b]}$$

Thus, $\mathcal{A}^{\dagger}(\alpha)$ defines or generates a two-dimensional Taylor expansion when it acts on a function of θ and π. In particular, if we take the "phase space" representation of the ground or vacuum state $|\theta\pi\rangle$ as the product of two Gaussians (see Equations [I-10a] and [I-29]), then $\mathcal{A}^{\dagger}(\alpha)|\theta\pi\rangle$ represents a shift or displacement of this "phase space" representation -- *i.e.*

$$\langle\theta\pi|\alpha\rangle = \langle\theta\pi|\mathcal{A}^{\dagger}(\alpha)|0\rangle = u_G(\theta - \alpha_r)\, u_G(\pi - \alpha_i) \qquad \text{[III-68]}$$

In light of Equation [II-23b], $|\alpha(t)\rangle = |\alpha \exp(-i\,\omega\, t)\rangle$ we can write

$$\langle\theta\pi|\alpha(t)\rangle = u_G\left(\theta - |\alpha|\cos(\omega\, t + \varphi)\right) u_G\left(\pi - |\alpha|\cos(\omega\, t + \varphi)\right) \qquad \text{[III-69]}$$

where $\alpha = |\alpha| \exp(i\,\varphi)$.

h. The diagonal coherent-state representation of the density operator (Glauber-Sudarshan P-representation):

It may be easily established that

$$\overline{\overline{\mathbf{1}}} = \frac{1}{\pi}\iint |\beta\rangle\langle\beta|\, d^2\beta = \iint |\beta\rangle\langle\beta|\, d\,\mathbf{Re}(\beta)\, d\,\mathbf{Im}(\beta) \qquad \text{[III-70]}$$

so that it seems quite reseasonable to look for a representation of the density matrix is the form

$$\overline{\overline{\rho}} = \iint P(\beta)\, |\beta\rangle\langle\beta|\, d^2\beta \qquad \text{[III-71]}$$

For a pure coherent state, P is clearly a two-dimensional delta function

Example 1 -- Coherent state

$$P(\beta) = \delta^{(2)}(\beta - \alpha) = \delta^{(1)}\left(\mathbf{Re}(\beta) - \mathbf{Re}(\alpha)\right)\delta^{(1)}\left(\mathbf{Im}(\beta) - \mathbf{Im}(\alpha)\right) \quad \text{[III-72]}$$

In general, using Equation [III-60] -- *i.e.*

$$\langle\beta|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha\beta^*\right] \quad \text{[III-60]}$$

we may find a simple procedure for finding the P-representation by writing

$$\begin{aligned}\langle-\alpha|\,\overline{\rho}\,|\alpha\rangle &= \iint P(\beta)\langle-\alpha|\beta\rangle\langle\beta|\alpha\rangle\, d^2\beta \\ &= \exp\left(-|\alpha|^2\right)\iint\left[P(\beta)\exp\left(-|\beta|^2\right)\right]\exp\left[\alpha\beta^* - \beta\alpha^*\right] d^2\beta\end{aligned} \quad \text{[III-73]}$$

Thus, $\langle-\alpha|\,\overline{\rho}\,|\alpha\rangle\exp\left(-|\alpha|^2\right)$ is the two-dimensional Fourier transform 0f the function $P(\beta)\exp\left(-|\beta|^2\right)$ and we may write

$$P(\beta) = \frac{1}{\pi^2}\exp\left(|\beta|^2\right)\iint\langle-\alpha|\,\overline{\rho}\,|\alpha\rangle\exp\left(|\alpha|^2\right)\exp\left[-\alpha\beta^* + \beta\alpha^*\right] d^2\alpha \quad \text{[III-74]}$$

As a second example, consider a thermal radiation field described by a canonical ensemble

$$\overline{\rho} = \frac{\exp\left(-\mathcal{H}/k_B T\right)}{\mathrm{Tr}\left[\exp\left(-\mathcal{H}/k_B T\right)\right]} \quad \text{[III-75]}$$

where $\mathcal{H} = \hbar\omega\left\{a^\dagger a + \frac{1}{2}\right\}$. Thus,

$$\overline{\rho} = \sum_n\left[1 - \exp\left(-\frac{\hbar\omega}{k_B T}\right)\right]\exp\left(-\frac{n\,\hbar\omega}{k_B T}\right)|n\rangle\langle n| \quad \text{[III-76]}$$

and $$\langle n \rangle = \mathrm{Tr}\left[\overline{\rho}\, a^{\dagger} a \right] = \sum_n \left[\exp\left(\frac{\hbar\omega}{k_B T} \right) - 1 \right]^{-1} \qquad \text{[III-77]}$$

so that $$\overline{\rho} = \sum_n \frac{\langle n \rangle^n}{\left(1 + \langle n \rangle\right)^{n+1}} \left| n \right\rangle \left\langle n \right| \qquad \text{[III-78]}$$

Thus, we can write $$\left\langle n \right| \overline{\rho} \left| n \right\rangle = \frac{\langle n \rangle^n}{\left(1 + \langle n \rangle\right)^{n+1}} \qquad \text{[III-79]}$$

$$\left\langle -\alpha \right| \overline{\rho} \left| \alpha \right\rangle = \sum_n \frac{\langle n \rangle^n}{\left(1 + \langle n \rangle\right)^{n+1}} \left\langle -\alpha | n \right\rangle \left\langle n | \alpha \right\rangle$$

and $$= \frac{\exp\left(-|\alpha|^2\right)}{1 + \langle n \rangle} \sum_n \frac{\left(-|\alpha|^2\right)^n}{n!} \left[\frac{\langle n \rangle}{\left(1 + \langle n \rangle\right)} \right]^n \qquad \text{[III-80]}$$

$$= \frac{\exp\left(-|\alpha|^2\right)}{1 + \langle n \rangle} \exp\left[-|\alpha|^2 \Big/ \left(1 + \frac{1}{\langle n \rangle} \right) \right]$$

Finally, we see that

Example 2 -- Thermal radiation - a chaotic state

$$P(\alpha) = \frac{\exp\left(|\alpha|^2 \right)}{\pi^2 \left(1 + \langle n \rangle\right)} \iint \exp\left[-|\beta|^2 \Big/ \left(1 + \frac{1}{\langle n \rangle} \right) \right] \exp\left(-\beta\alpha^* + \alpha\beta^* \right) d^2\beta$$
$$= \frac{1}{\pi \langle n \rangle} \exp\left(-|\alpha|^2 / \langle n \rangle \right) \qquad \text{[III-81]}$$

As a third example, consider Fock or number state. From Equation [III-55] we see that

$$\langle -\alpha | \overline{\rho} | \alpha \rangle = \langle -\alpha | n \rangle \langle n | \alpha \rangle = \frac{\exp\left(-|\alpha|^2\right)}{n!}\left(-|\alpha|^2\right)^n \qquad \text{[III-82a]}$$

and

$$\begin{aligned} P(\beta) &= \frac{1}{n!}\frac{1}{\pi^2}\exp\left(|\beta|^2\right)\iint \left(-|\alpha|^2\right)^n \exp\left[-\alpha\beta + \beta\alpha\right] d^2\alpha \\ &= \frac{\exp\left(|\beta|^2\right)}{n!}\frac{\partial^{2n}}{\partial\beta^{*n}\partial\beta^{n}}\frac{1}{\pi^2}\iint \exp\left[-\alpha\beta + \beta\alpha\right] d^2\alpha \end{aligned} \qquad \text{[III-82b]}$$

so that

Example 3 -- Pure Fock or number state

$$P(\beta) = \frac{\exp\left(|\beta|^2\right)}{n!}\frac{\partial^{2n}}{\partial\beta^{*n}\partial\beta^{n}}\delta^{(2)}(\beta) \qquad \text{[III-82b]}$$

i. **The Glauber-Sudarshan-Klauder "optical equivalence" theorem:**

Suppose we have some "normally ordered" function

$$f^{(N)}\left(a, a^{\dagger}\right) = \sum_n \sum_m c_{nm}\, a^{\dagger n} a^{m} \qquad \text{[III-83]}$$

The expectation value is given by

$$\left\langle f^{(N)}\left(a, a^{\dagger}\right)\right\rangle = \mathrm{Tr}\left[\overline{\rho}\, f^{(N)}\left(a, a^{\dagger}\right)\right] \qquad \text{[III-84]}$$

Using Equation [III-71] we see that

$$\left\langle f^{(N)}\left(a, a^{\dagger}\right)\right\rangle = \mathrm{Tr}\left[\iint P(\alpha) \sum_n \sum_m c_{nm} \,|\alpha\rangle\langle\alpha|\, a^{\dagger n} a^m \, d^2\alpha\right]$$

$$= \iint P(\alpha) \sum_n \sum_m c_{nm} \langle\alpha|\, a^{\dagger n} a^m \,|\alpha\rangle \, d^2\alpha \qquad \text{[III-85a]}$$

$$= \iint P(\alpha) \sum_n \sum_m c_{nm}\, \alpha^{*n} \alpha^m \, d^2\alpha$$

or, finally, the **“optical equivalence” theorem**

$$\left\langle f^{(N)}\left(a, a^{\dagger}\right)\right\rangle = \iint P(\alpha)\, f^{(N)}\left(\alpha, \alpha^*\right) \qquad \text{[III-85b]}$$

j. **The Uncertainty Relationship for $\{\theta, \pi\}$:**

Since $\left[a, a^{\dagger}\right] = 1$ we see from Equation [III-64a] that

$$\left\langle \Delta\theta^2 \right\rangle = \left\langle \theta^2 \right\rangle - \left\langle \theta \right\rangle^2$$

$$= \left\langle a^{\dagger} a^{\dagger} \right\rangle \exp[2\, i\, \gamma] + \left\langle a \; a \right\rangle \exp[-2\, i\, \gamma] + \left\langle a^{\dagger} \; a \right\rangle + \left\langle a \; a^{\dagger} \right\rangle$$

$$- \left\langle a^{\dagger} \right\rangle^2 \exp[2\, i\, \gamma] - \left\langle a \right\rangle^2 \exp[2\, i\, \gamma] - 2\left\langle a^{\dagger} \right\rangle \left\langle a \right\rangle \qquad \text{[III-86]}$$

$$= \left\langle : \Delta\theta^2 : \right\rangle + 1$$

where $\langle : \mathbf{A} : \rangle$ symbollizes the normally ordered expectation value of the operator $\mathbf{A}$. From Equation [III-85b]

$$\left\langle : \Delta\theta^2 \left(a, a^{\dagger}\right) : \right\rangle = \iint P(\alpha)\, \Delta\theta^2\left(\alpha, \alpha^*\right) d^2\alpha \qquad \text{[III-87]}$$

$$\left\langle :\Delta\theta^2\left(a, a^\dagger\right): \right\rangle = \iint P(\alpha)\left[\Delta\alpha^* \exp(i\gamma) + \Delta\alpha \exp(-i\gamma)\right]^2 d^2\alpha \qquad [\text{ III-88 }]$$

If we choose γ (and $P(\alpha)$) such that $\left\langle :\Delta\theta^2\left(a, a^\dagger\right): \right\rangle < 0$, then $\left\langle \Delta\theta^2 \right\rangle > 1$ and $\left\langle \Delta\pi^2 \right\rangle > 1$ (**squeezed states**)!